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Nondeterministic "Possibilistic" Approaches for Structural Analysis and Optimal Design

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The development of methods to take into account uncertainties in structural analysis and in design optimization is attracting both the scientific and the industrial communities. In this domain possibilistic methods in which uncertainties are defined by fuzzy numbers appear as an alternative to the classical probabilistic methods such as the Monte Carlo simulations or the stochastic finite element method. The principal difficulty of possibilistic methods is that they lead to the resolution of systems of equations in which the coefficients are defined by intervals. Several approaches for the direct solution of such interval linear equations systems are presented and compared. The vertex method is taken as reference. It is shown that the problems that are solved are mathematically different according to the method used. For static linear analysis a cost-effective iterative solution of the vertex is proposed. It is based on Neumann series expansions and solves an optimization subproblem to compute extrema of the structural responses. The effectiveness of the method is illustrated by the solution of truss problems, classical in the optimization literature. Extension of the method to inverse problems of design is also considered.

Introduction

TODAY, nondeterministic problems are receiving a fast-growing interest from both the scientific community and various industrial sectors. It is clear that with computational methods for structural analysis, now mature and widely used, the next challenge is to extend these methods to cope with uncertainties and make possible estimation of their effects. Most of the time uncertainties will be more or less directly related to product manufacturing and assembly conditions. So, all of the data appearing in the definition of a finite element model can be scattered. This is the case for geometrical dimensions because of the variability of structural sizes and for the physical data, like the sheet metal thickness or the cross sections and inertia properties of beams. This is also true for the material data because of nonhomogeneity of the microstructural properties in relation with both material and manufacturing aspects, or at macroscopic level, the Young's modulus, the yield stresses, the fatigue life equations. The boundary conditions like loads or prescribed displacements are also scattered because of the variation of their intensity or their time history. Finally, the existence of initial defects resulting from the manufacturing process has an impact both on the failure mechanisms and on the way they are computed.

Apart from the well-known discretization error, the modeling process itself introduces approximations, through the choice of the type of model or the type of analysis and in the idealization of shapes, dimensions, material properties, and loads. These approximations can also be interpreted in some way as uncertainties. From the computational point of view, the scattering of modeling data will affect the coefficients in the matrices defining the finite element model, like the stiffness matrix, the mass matrix, and the loads. This will in turn lead to the scattering of the structural responses.

When dealing with data uncertainties, several objectives are possible. The first one is clearly to get sensitivity information and quantify how the structural responses will be influenced by the data uncertainties. Under stochastic phenomena, reliability and estimation of failure probabilities are clearly a second target. However, the main benefit from nondeterministic computational methods is perhaps for design, either in helping the designer in finely tuning process parameters to achieve a prescribed level of reliability or to design structures as insensitive as possible to uncertainties. This is usually known as robust design.

There exist today several techniques to compute the uncertainties of structural responses. The best known are the now classical Monte Carlo simulations (MCS) and the stochastic finite element method (SFEM). A third class of methods, which uses the concept of fuzzy numbers to represent uncertainties and arithmetic of intervals to compute structural responses, is considered in this paper. The latter differs from the classical probabilistic approaches in the sense that, instead of computing statistical distributions of structural responses, distributions of possibilities are obtained. For that reason they are known as possibilistic approaches. Using interval arithmetic, these methods have been introduced recently[1,2] in structural engineering.

The paper first will be devoted to the review of different possible direct algorithms to solve interval linear equations systems. The vertex method,[3] which requires for each level of membership the solution of $2n$ real systems, for n uncertain parameters, will be used as reference. Using an example, we will show that the mathematical problems solved in the direct and the vertex methods are not the same. Then, in the framework of static linear analysis, an original cost-effective method will be proposed to compute the vertex solution. The method uses Neumann-series expansions of the structural responses and solves an optimization subproblem to find the extrema of the responses. Truss problems, classical in the structural optimization literature, will be used to illustrate the method. The number of uncertain variables in these problems will preclude them from the vertex method. Finally, the extension of possibilistic methods to inverse design problems will be introduced.

Modeling Uncertainties

Several examples can be found in literature where fuzzy numbers cope with structural uncertainties. For instance, fuzzy numbers are used to evaluate the sensitivities of structures to uncertainties of prescribed displacements.[4] The development of fuzzy approaches and interval arithmetics for design[5] was also used to formulate a design problem.[6] A fuzzy number combines two ideas. The first one is the concept of interval of confidence, which defines the variation range of the variable. The second is the concept of degree of membership,

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which can be better understood as a level of satisfaction: the higher the level of satisfaction is, the smaller the interval of variation is. A fuzzy number $\mu(x)$ can thus be defined as a set of intervals of the parameter x, which are ordered for increasing values of the satisfaction degree α, which describes a membership degree, ranging from 0 to 1. At the level of satisfaction α, the variation domain of the variable x is given by the interval $[\underline{I}_\alpha; \bar{I}_\alpha]$. The two main properties of fuzzy numbers are normality [meaning that $\exists x \in \Re : \mu(x) = 1$] and convexity, which yield to the nesting property:

$$\text{if } \alpha_1 < \alpha_2 \text{ then } \underline{I}_{\alpha_1} \le \underline{I}_{\alpha_2}, \qquad \bar{I}_{\alpha_1} \ge \bar{I}_{\alpha_2}$$

In practice, the fuzzy response of a structure is computed in three steps. First, the fuzzy numbers describing the parameters uncertainties are sampled for different degrees of membership α giving for each of them a set of intervals. This is known as fuzzification. Second, the finite element equilibrium equations are solved at each level, leading to the corresponding variation intervals of the structural responses. Finally, putting together for each structural response, the intervals related to the different degrees of membership allow the fuzzy response to be rebuilt.

As far as computation for a given degree of membership is concerned and as uncertainties do not only affect the boundary conditions at the right-hand side but also the coefficients of the system through the geometrical dimensions, the material and physical properties, the difficulty is clearly to solve discretized interval equilibrium equations.

Neither modeling of uncertainties nor interpretation of fuzzy results will be considered here. The emphasis will be placed on the computation of the uncertain structural responses for one prescribed level of membership. So, the concept of fuzzy number will no longer appear, and only mathematics of intervals will be considered. A second assumption concerns the type of analysis. Only static linear analysis of structures is envisaged hereafter with the goal to predict the scattering of either flexibility, stresses, or loads. Finally, interpretation and comparison of possibilistic results in terms of probabilities will not be addressed.

Mathematics of Intervals

Arithmetic of Intervals[7]

Let the symbol $\hat{\cdot}$ identify intervals and $\bar{\cdot}$ mean values. An interval $\hat{a}$ is a subset of $\Re$ defined by

$$\hat{a} = [a_1; a_2] = \{x : a_1 \le x \le a_2; a_1, a_2 \in \Re\}$$

The mathematical operations are defined as follows:

Addition:

$$\hat{a} + \hat{b} = [a_1 + b_1; a_2 + b_2]$$

Subtraction:

$$\hat{a} - \hat{b} = [a_1 - b_2; a_2 - b_1]$$

Multiplication:

$$\hat{a} \times \hat{c} = [\min C; \max C] \quad \text{with} \quad C = \{a_1 b_1, a_1 b_2, a_2 b_1, a_2 b_2\}$$

Division:

$$\hat{a}/\hat{b} = [a_1; a_2] \times [1/b_2; 1/b_1] \quad \text{with} \quad 0 \notin [b_1; b_2]$$

Intervals present the usual mathematical properties (commutativity, associativity, ...) and more specific ones such as subdistributivity and monotone inclusion: Let $\hat{a}_1, \hat{b}_1, \hat{a}_2, \hat{b}_2$ be four intervals, with $\hat{a}_i \subseteq \hat{b}_i$ for $i = 1, 2$ then

$$\hat{a}_1 \bullet \hat{a}_2 \subseteq \hat{b}_1 \bullet \hat{b}_2 \forall \bullet \in \{+, -, x, /\}$$

Extension to interval matrices is direct with $\hat{A} = \hat{A}_{ij} = [\underline{a}_{ij}; \bar{a}_{ij}]$, $(1 \le i \le n, 1 \le j \le n)$ as the $n \times n$ interval matrix.

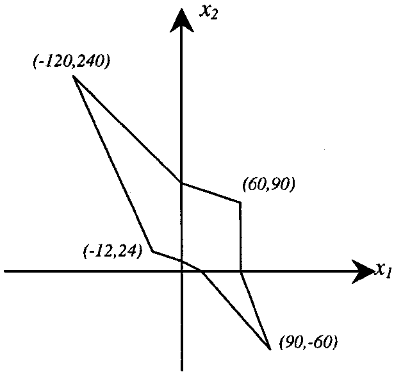

Fig. 1 Representation of the solution set.

Interval Linear Equations Systems

Writing an interval linear system of equations $\hat{A}\hat{x} = \hat{b}$ as the family of equations:

$$Ax = b$$

with $A \in \hat{A}$ ($\Re^{n \times n}$ interval matrix) and $b \in \hat{b}$ ($\Re^n$ interval vector), the solution set will be defined as

$$\sum(\hat{A}, \hat{b}) = \{x \in \Re^n / Ax = b, A \in \hat{A} \quad \text{and} \quad b \in \hat{b}\} \qquad (1)$$

One can see that the structure of $\sum(\hat{A}, \hat{b})$ is complex. In practice, it will not be possible to compute the exact solution set, and it is not necessarily an interval vector; only estimators of the solution set will be obtained. The difficulty encountered to express this set is obvious[1,2] even for small systems with reduced input variables magnitudes. A simple example taken from Hansen's works[8] will help fix ideas.

Considering the system

$$\begin{bmatrix} [2;3] & [0;1] \\ [1;2] & [2;3] \end{bmatrix} \begin{Bmatrix} x_1 \\ x_2 \end{Bmatrix} = \begin{Bmatrix} [0;120] \\ [60;240] \end{Bmatrix}$$

Figure 1 represents the exact solution set.

Methods Based on Arithmetics of Intervals

The solution of the interval linear equations system will thus consist in finding an estimation of the solutions set. In practice, the estimator can be either optimistic or pessimistic according to the way the system is considered, either in its implicit or explicit form. Two different types of approaches are thus available.

Implicit Formulation

The implicit formulation corresponds to the optimistic approximation of the solution set. The interval linear equations system $\hat{A}\hat{x} = \hat{b}$ is written in the following way:

$$\begin{pmatrix} [\underline{a}_{11}, \bar{a}_{11}] & \cdots & [\underline{a}_{1n}, \bar{a}_{1n}] \\ [\underline{a}_{21}, \bar{a}_{21}] & \cdots & [\underline{a}_{2n}, \bar{a}_{2n}] \\ \vdots & \ddots & \vdots \\ [\underline{a}_{n1}, \bar{a}_{n1}] & \cdots & [\underline{a}_{nn}, \bar{a}_{nn}] \end{pmatrix} \begin{pmatrix} [\underline{x}_1, \bar{x}_1] \\ [\underline{x}_2, \bar{x}_2] \\ \vdots \\ [\underline{x}_n, \bar{x}_n] \end{pmatrix} = \begin{pmatrix} [\underline{b}_1, \bar{b}_1] \\ [\underline{b}_2, \bar{b}_2] \\ \vdots \\ [\underline{b}_n, \bar{b}_n] \end{pmatrix}$$

meaning that the estimator of the solution set is sought as the interval vector $\hat{x}$, which allows matching uncertainties of the left-hand side of the equations to the uncertainties on the right-hand side $\hat{b}$. Provided that the system is compatible, i.e., uncertainties on $\hat{b}$ are greater or equal to the uncertainties resulting from the product of $\hat{A}$ and $\hat{x}$ (in the sense of the arithmetic of intervals), the solutions are known as optimistic as the intervals of $\hat{x}$ components can be underestimated.

To compute optimistic solutions, an original formulation consists in solving the following large sparse linear mathematical programming problem:

$$\max \min(\bar{x}_i - \underline{x}_i)$$

subject to

$$\underline{x}_i \le \bar{x}_i \qquad i = 1, n$$

$$\begin{bmatrix} \sum_i \underline{u}_{ij} = \underline{b}_j \\ \sum_i \bar{u}_{ij} = \bar{b}_j \end{bmatrix} \qquad j = 1, n$$

$$\begin{bmatrix} \underline{u}_{ij} \le \underline{a}_{ij}\underline{x}_i & \bar{u}_{ij} \ge \underline{a}_{ij}\underline{x}_i \\ \underline{u}_{ij} \le \underline{a}_{ij}\bar{x}_i & \bar{u}_{ij} \ge \underline{a}_{ij}\bar{x}_i \\ \underline{u}_{ij} \le \bar{a}_{ij}\underline{x}_i & \bar{u}_{ij} \ge \bar{a}_{ij}\underline{x}_i \\ \underline{u}_{ij} \le \bar{a}_{ij}\bar{x}_i & \bar{u}_{ij} \ge \bar{a}_{ij}\bar{x}_i \end{bmatrix} \qquad i, j = 1, n$$

where the slack variables u_{ij} are introduced to enforce the intervals multiplication rules. Additionally, a perturbation strategy consisting in solving the initial system perturbed around mean values is used to cope with compatibility problems[9]:

Introducing $\hat{A} = \tilde{A} + \Delta\hat{A}$, $\hat{x} = \tilde{x} + \Delta\hat{x}$, $\hat{b} = \tilde{b} + \Delta\hat{b}$, one has

$$[\hat{A}][\Delta\hat{x}] \subseteq [\Delta\hat{b}] - [\Delta\hat{A}]\tilde{x}$$

with respectively $\hat{A}$, $\tilde{A}$, and $\Delta\hat{A}$, the $\Re^{n\times n}$ intervals matrix, the mean value, and the perturbation.

Explicit Formulation

The explicit formulation of the interval linear equations system $\hat{x} = \hat{A}^{-1}\hat{b}$ will lead to a pessimistic estimation of the solution set, as computing $\hat{x}$ can be seen as an error propagation process. An overestimation of the interval components of $\hat{x}$ is obtained, sometimes unbounded.

Hansen Algorithm

One of the most popular explicit direct algorithms is the Hansen algorithm. Starting from the fact that the zero value cannot be included in an interval at the denominator of a division, iterative algorithms of the Gauss–Seidel or Jacobi family have to be used as a basis to solve interval linear equations systems. Iterates in the Gauss–Seidel algorithm are given by

$$\hat{x}_i^{(k+1)} = \frac{1}{\hat{a}_{ii}}\left(\hat{b}_i - \sum_{j=1}^{i-1}\hat{a}_{ij}\hat{x}_j^{(k+1)} + \sum_{j=i+1}^{n}\hat{a}_{ij}\hat{x}_j^{(k)}\right) \tag{2}$$

where, at the contrary of the Gauss algorithm, the $\hat{a}_{ij}$ are known and positive if $\hat{A}$ corresponds to the stiffness matrix. All terms in the preceding equation are intervals. Two modifications were introduced by Hansen[8] to limit occurrence of unbounded solutions in the basic algorithm: the first one is a midpoint inverse preconditioning of the system and the second an intersection strategy between two successive iterates.

Midpoint Inverse Preconditioning

Hansen's algorithm[8,10] solves the preconditioned system

$$\hat{M}\hat{x} = \hat{r}$$

with $\hat{M} = \tilde{A}^{-1}\hat{A}$, $\hat{r} = \tilde{A}^{-1}\hat{b}$, $\tilde{A} = (\bar{A} + \underline{A})/2$.

Intersections

Using the Gauss–Seidel's algorithm for the solution of the system $\hat{M}\hat{x} = \hat{r}$, one has at iteration k

$$\hat{x}_i^{(k+1)} = \frac{1}{\hat{m}_{ii}}\left(\hat{r}_i - \sum_{j=1}^{i-1}\hat{m}_{ij}\hat{x}_j^{(k+1)} + \sum_{j=i+1}^{n}\hat{m}_{ij}\hat{x}_j^{(k)}\right)$$

with

$$\hat{x}_i^{(k+1)} = \hat{x}_i^{(k)} \cap \hat{y}_i^{(k+1)}$$

and

$$\hat{y}_i^{(k+1)} = \frac{1}{\hat{m}_{ii}}\left(\hat{r}_i - \sum_{j=1}^{i-1}\hat{m}_{ij}\hat{x}_j^{(k+1)} + \sum_{j=i+1}^{n}\hat{m}_{ij}\hat{x}_j^{(k)}\right)$$

According to a pessimistic estimation of the solutions, the intersection strategy helps in minimizing errors accumulation and avoiding unbounded solutions. Rohn's algorithm[11] gives further extensions to the basic Hansen's algorithm.

Vertex Solution

For all of the problems considered in the experimental stage, and when possible, the vertex method[12] was used as reference. The vertex method consists in successively exploring the $2n$ combinations of the extreme values of the design variables and taking the corresponding extreme values of the structural responses. The vertex method implicitly assumes a monotone dependency between the structural responses and the design variables.

Coming back to the simple example just introduced,

$$\begin{bmatrix} [2;\ 3] & [0;\ 1] \\ [1;\ 2] & [2;\ 3] \end{bmatrix}\begin{Bmatrix} x_1 \\ x_2 \end{Bmatrix} = \begin{Bmatrix} [0;\ 120] \\ [60;\ 240] \end{Bmatrix}$$

Tables 1 and 2 give values of the unknowns x and the recomputed right-hand sides for the four methods: the vertex method, the Hansen algorithm, the classical Gauss–Seidel algorithm, and finally the implicit formulation (respectively subscripted v, h, g, and s).

Figure 2 shows representations of various estimators of the solution set (dashed lines) in comparison with the exact solution set (continuous line).

From the solution of some small-to medium-sized problems, the following conclusions arise:

1) Perturbation and preconditioning enlarge the estimations of the solution set.

2) The implicit method enforces the left-hand side of the linear equations system to the (perturbed) right-hand side. This is not the case in the explicit approaches where the recomputed right-hand side are larger that the prescribed ones.

3) The perturbation strategy is essential to ensure feasibility in the optimistic formulation.

4) The preconditioning strategy is essential to ensure robustness in the pessimistic algorithms.

Table 1 Unknown intervals x

Method	x_1	x_2
q_v	[−120; 90]	[−60; 240]
q_g	[−120; 90]	[−60; 240]
q_h	[−130.2; 167.7]	[−104.4; 267.2]
q_s	[0; 22.5]	[30; 52.5]

Table 2 Recomputed right-hand sides

Method	b_1	b_2
q_v	[−420; 510]	[−420; 900]
q_g	[−420; 510]	[−420; 900]
q_h	[−495; 770.3]	[−537.7; 1137.1]
q_s	[0; 120]	[60; 202.5]

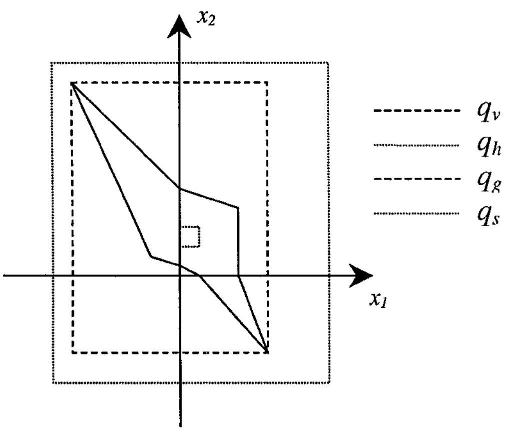

Fig. 2 Comparison of the different solutions sets.

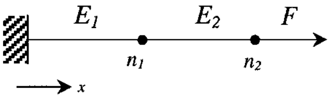

Fig. 3 Two-element bar.

5) The vertex solution always appears to be the reference solution.

6) Hansen's algorithm and its Rohn's variant provide solution sets greater or equal to those provided by the vertex method with a quite good robustness.

7) Emphasise is also placed on the fact that prior to the solution of intervals equations systems extreme values of the systems coefficients have to be computed.

Finally, the following example will fix ideas on the differences between the direct and vertex methods. Figure 3 represents a bar modeled by two elements.

For simplicity, the lengths, the cross sections, the load, and the Young's modulus of the two elements are all taken equal to unity. The uncertain parameter is the Young's modulus $\hat{E}_2$ in the second bar, which is allowed to vary in the interval $\hat{E}_2 = [0.9; 1.1]$. The equilibrium equations are given by

$$\begin{bmatrix} 1+\hat{E}_2 & -\hat{E}_2 \\ -\hat{E}_2 & \hat{E}_2 \end{bmatrix} \begin{Bmatrix} q_1 \\ \hat{q}_2 \end{Bmatrix} = \begin{Bmatrix} 0 \\ 1 \end{Bmatrix}$$

so

$$\begin{Bmatrix} q_1 \\ \hat{q}_2 \end{Bmatrix} = \begin{Bmatrix} 1 \\ (1+\hat{E}_2)/\hat{E}_2 \end{Bmatrix}$$

The substitution of the values 0.9 and 1.1 leads to the vertex solution: $\hat{q}_2 = [1.909; 2.111]$. However, because of the arithmetic of intervals, the direct solution will be given by either $\hat{q}_2 = [1.727; 2.333]$ or $\hat{q}_2 = [1.909; 2.111]$, accordingly $\hat{q}_2$ is computed as $\hat{q}_2 = (1+\hat{E}_2)/\hat{E}_2$ or $\hat{q}_2 = 1 + (1/\hat{E}_2)$. This illustrates the dependency phenomenon, which is one origin of the difficulties in the direct methods.

Neumann Approximate Vertex Solution

As the vertex method remains the most robust approach and leads most of the time to the best estimation of the solutions set, a way was searched to improve its efficiency. In this section the solution of the vertex problem is built using Neumann-series expansion. Assuming a linear static analysis problem, it is formally possible to express the intervals including the variations of the stiffness matrix and of the loads vector caused by the variations of the uncertain parameters d_i as

$$\hat{K} = [\underline{K}; \bar{K}] = \tilde{K} + \sum_{i=1}^{n_1} \mu_i \Delta K_i(d_i) \tag{3}$$

with $\underline{K}$ the lower bound of the intervals for the stiffness matrix, $\bar{K}$ the upper bound, $\tilde{K}$ the mean value of the stiffness matrix, and ΔK_i the stiffness matrix perturbations corresponding to the ith variable. The μ_i is a vector of scalars equal to -1 or $+1$ according to $\underline{K}$ or $\bar{K}$.

Similar to Eq. (2), one can write for the loads

$$\hat{g} = [\underline{g}; \bar{g}] = \tilde{g} + \sum_{i=n_1+1}^{n} \mu_i \Delta g_i(d_i) \tag{4}$$

Consequently, considering without loss of generality only uncertainties on the stiffness one can write successively

$$Kq = g$$

$$q = \left(\tilde{K} + \sum_{i=1}^{n} \mu_i \Delta K_i\right)^{-1} g$$

$$q = \sum_{r=0}^{\infty} \left(-\sum_{i=1}^{n} \mu_i \tilde{K}^{-1} \Delta K_i\right)^{r} q_0 \quad \text{with} \quad q_0 = \tilde{K}^{-1} g$$

or recursively

$$q^{(k+1)} = q^0 + \sum_{i=1}^{n} \mu_i \tilde{K}^{-1} \Delta K_i q^{(k)} \tag{5}$$

The ΔK_i are the n perturbations of the stiffness matrix, either the ith term in $\Delta K = (\bar{K} - \underline{K})/2$, in which case the extreme values of the different terms of K have to be a priori computed, or given by $\Delta K_i = \partial K/\partial d_i|_0 \, \delta d_i$ if the stiffness matrix is linear in the design variables d_i.

Considering now a design criteria $c(q)$, e.g., $c = b^T q$, the signs of the μ_i in Eq. (1) will be selected so as to minimize or maximize c, depending on which extrema ($\underline{c}$ or $\bar{c}$) is searched. A closed-form solution is immediate when the stiffness matrix is linear in the uncertain parameters. In this case the design criteria are monotone functions of the μ_i and the vertex method is rigorous. When this is not the case, the best strategy—yet to explore—would consist in expanding the perturbations in series, leading to the solution of an optimization subproblem.

Numerical Examples

The test cases are selected among the well-known truss problems of the structural optimization literature.[13] The 10-bar truss, the 25-bar truss transmission tower, and the 720-bar four-levels space truss problems are successively considered, leading to a number of uncertain design variables with which the vertex method cannot cope. Uncertainties affect the Young's modulus of the bars with a magnitude of 20%. A variable per bar is assumed. The results consist of the scattering of the values of flexibility at prescribed nodes in percentage of the values of the mean displacement. The signs of the μ_i are also given. The Neumann approximate vertex solution (NAVS) method reproduces results from the vertex method and thus overcomes difficulties encountered with the direct methods. On the other hand, the efficiency of the method opens the door to the solution of reasonable size problems.

Simple Cantilever 10-Bar Truss

A simple example is given by the 10-bar truss represented in Fig. 4. In this problem the mean characteristics are $E = 70{,}000$ MPa, $a = 1290$ mm^2, and $F_1 = F_2 = 10^5$ N. For the preceding values the mean displacements are

$$\text{for the third node} \begin{cases} \text{along } x\text{:} & u_3 = -1.464 \text{ mm} \\ \text{along } y\text{:} & v_3 = -6.059 \text{ mm} \end{cases}$$

One can now envisaged the evaluation of the effect of an uncertainty of $\pm 10\%$ on the values of the bars Young's modulus. The results for both the vertex and the NAVS methods are presented in Table 3.

According to the number of design variables, the vertex solution requires 1024 finite element analysis whereas the NAVS solution requires four computations, two for each component of displacements with a convergence achieved in less than five iterations. Table 4 gives the signs of the μ_i for each design variable and for each component

Table 3 Uncertainties on Young's modulus

%	u_3	v_3
Vertex	[−14.9; 12.4]	[−11.1; 9.11]
NAVS	[−14.9; 12.4]	[−11.1; 9.11]

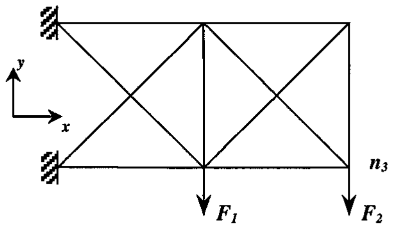

Fig. 4 Ten-bar truss problem.

Table 4 Signs of the μ_i

μ_i	$\underline{u}_3$	$\bar{u}_3$	$\underline{v}_3$	$\bar{v}_3$
E_1	−	+	−	+
E_2	−	+	−	+
E_3	−	+	−	+
E_4	−	+	−	+
E_5	−	+	−	+
E_6	−	+	−	+
E_7	+	−	−	+
E_8	−	+	−	+
E_9	+	−	−	+
E_{10}	−	+	+	−

Table 5 Uncertainties on loads and Young's modulus

%	u_3	v_3
Vertex	[−26.4; 21.2]	[−22.3; 18.2]
NAVS	[−26.4; 21.2]	[−22.3; 18.2]

Table 6 Signs of the μ_i

μ_i	$\underline{u}_3$	$\bar{u}_3$	$\underline{v}_3$	$\bar{v}_3$
E_1	−	+	−	+
E_2	−	+	−	+
E_3	−	+	−	+
E_4	−	+	−	+
E_5	−	+	−	+
E_6	−	+	−	+
E_7	+	−	−	+
E_8	−	+	−	+
E_9	+	−	−	+
E_{10}	−	+	+	−
F_1	−	+	−	+
F_2	−	+	−	+

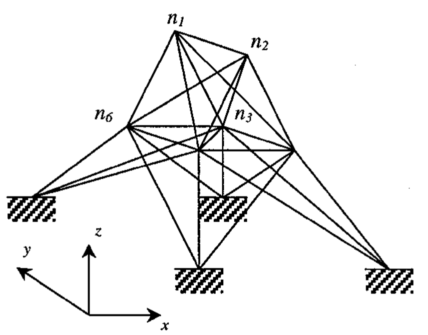

Fig. 5 Twenty-five-bar truss problem.

of displacement (Note that as displacements are negative, the minimum values will be obtained for the smallest Young's modulus.).

The fact that the signs are opposite for the minimum and maximum values of a same component of displacement confirms the monotone character of the functions. Half of the computations would have been necessary in this case. The same problem was solved adding uncertainties of ±10% on the two loads (Tables 5 and 6).

Twenty-Five-Bar Truss Transmission Tower

This example, very popular in the structural optimization literature, is defined in Fig. 5. One load case is considered and defined in Table 7. The physical properties are the same as for the 10-bar truss.

Uncertainties are once again on the material properties. Results are summarized in Table 8, giving the mean displacement and minimum and maximum possible values (in both millimeters and percentage) for the three components of displacements at nodes 1 and 6.

Scatterings on small values of displacements are proportionally larger. Convergence is achieved just as for the 10-bar truss problem.

Table 7 Load case

	Load component, N		
Node	x	y	z
1	10,000	100,000	−50,000
2	0	100,000	−50,000
3	5,000	0	0
6	5,000	0	0

Table 8 Twenty-five-bar truss transmission tower

Response	$\tilde{u}$, mm	$\underline{u}$, mm	$\bar{u}$, mm	$\underline{u}$, %	$\bar{u}$, %
u_1	−0.391	1.291	0.445	−187.9	189.6
v_1	7.807	9.582	8.606	−9.3	11.3
w_1	−0.844	−0.097	−0.465	−81.3	79.1
u_6	−0.064	0.364	0.147	−143.3	147.1
v_6	0.375	0.762	0.558	−32.6	36.6
w_6	1.281	1.880	1.555	−17.6	20.9

Table 9 Load case

	Load component, N		
Node	x	y	z
17	50,000	50,000	−50,000

Table 10 Seventy-two-bar truss transmission tower

Response	$\tilde{u}$, mm	$\underline{u}$, mm	$\bar{u}$, mm	$\underline{u}$, %	$\bar{u}$, %
u_{17}	0.618	0.819	0.710	−13.0	15.3
v_{17}	0.618	0.819	0.710	−13.0	15.3
w_{17}	0.059	0.139	0.098	−39.6	42.5

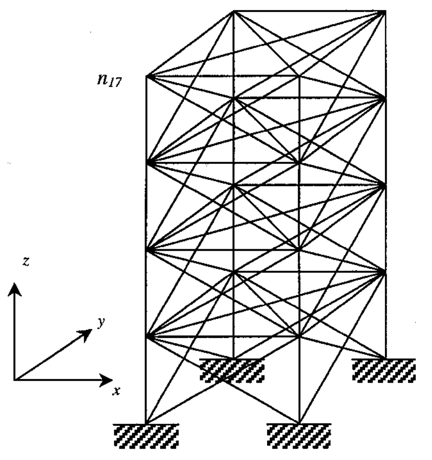

Fig. 6 Seventy-two-bar truss problem.

Vertex solution requires more than 33 millions of finite element analyses and NAVS requires only two per observed displacement!

Four-Level 72-Bar Space Truss

This problem has been investigated by many authors in structural optimization. It is defined in Fig. 6, and one load case is considered (Table 9). The physical properties are again the same.

This example clearly shows an example of robust design for u_{17} and v_{17} values (Table 10). As expected, vertical displacement is more scattered than the others are. Output uncertainties do not depend on the number of inputs overall on the design. To perform this example, vertex method needs 4.7×10^{21} calculations. Therefore, our method allows solving, with a reliable process, problems strictly impossible to treat with the vertex method. The maximum value of u_{17} for example is obtained for positive μ_i in bar 1, 4, 11,12, 16, 16, 19, 22, 29, 30, 33, 34, 40, 47, 48, 54, 59, 65, 66, 67, 69, and 72. As a result, the scattering obtained by setting all the μ_i equal to $+1$ or -1 is smaller than this obtained by the NAVS method: [0.175; 0.214].

Inverse Design Problem

Now that a method is available to compute uncertainties on structural design criteria according to uncertainties on the design variables, an industrial frequent question is which tolerances should be specified on the design variables so as to achieve prescribed uncertainties on the design criteria. Let us once more consider an example to fix idea. In the design of a flange joint, the designer will have to maintain pressure on the leak joint between extreme values to avoid leakage. As the pressure on the joint depends on design parameters, the problem will be to find for the latter parameters the intervals of variation that will make sure that the pressure is in the prescribed interval. Such a problem can be mathematically formulated as an inverse optimization problem:

$$\max\,(\min \delta d_i) \quad \text{or} \quad \max\,(\min \Delta K_i)$$

Subject to:

$$\max\,[c(q)] = \bar{c} \qquad \min\,[c(q)] = \underline{c}$$

where $\max[c(q)]$, $\min[c(q)]$, and their derivatives are computed as previously. Under the assumption of monotony of the vertex method, the confidence on the design variables could be transferred on the design criteria.

Conclusions

In this paper uncertainties in structural analysis are considered through so-called possibilistic formulations. Representing uncertainties on the design parameters by fuzzy numbers, different algorithms have been presented to compute intervals of variation of design criteria. The respective performances of the algorithms have been discussed. Putting aside the cost criteria, the vertex method appears to remain the most robust and most of the time gives the best approximations to the solutions set. An iterative approach has been proposed for building vertex solutions, using Neumann-series expansion and sensitivity analysis information. This approach, contrary to existing ones, allows the management of widely spread variables on large structural systems. Using this results, possibilistic distributions of design criteria can be built. Another possibility is to use this information to set, through the solution of inverse design problems, specifications on design parameters.

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